University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES

XX10190

Programming and discrete mathematics

Thursday 23rd January 2020
13:00 – 14:00
1 hour

Answer ALL questions from the paper in the WHITE booklet.

Use the YELLOW booklet for any rough work. This booklet will not be collected.

No calculators may be brought in and used.

PLEASE FILL IN THE DETAILS ON THE FRONT OF YOUR ANSWER BOOK/COVER AND SIGN IN THE SECTION ON THE RIGHT OF YOUR ANSWER BOOK/COVER, PEEL AWAY ADHESIVE STRIP AND SEAL.

TAKE CARE TO ENTER THE CORRECT CANDIDATE NUMBER AS DETAILED ON YOUR DESK LABEL.

DO NOT TURN OVER YOUR QUESTION PAPER UNTIL INSTRUCTED TO BY THE CHIEF INVIGILATOR.

Let $f(n)$ and $g(n)$ be two real valued functions defined on the integers.

(a) f is of the order at most g, written $f(n) = \mathcal{O}(g(n))$, if there exists positive numbers B and b, such that
$$|f(n)| \leq B|g(n)|,$$
for all $n \geq b$.

(b) f is of the order at least g, written $f(n) = \Omega(g(n))$, if there exists positive numbers A and a, such that
$$|f(n)| \geq A|g(n)|,$$
for all $n \geq a$.

(c) f is of order exactly g, written $f(n) = \Theta(g(n))$, if $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$.

1. Let
$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix},$$
where $\alpha \in \mathbb{R}$ is a constant such that $\alpha > 0$.

(a) Compute A^2, A^3, A^4. [3]

(b) Prove that
$$A^n = \begin{bmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{bmatrix} \qquad (\dagger)$$
for all integers $n \geq 1$. [10]

(c) Let
$$f(n) = \sum_{i=1}^{2}\sum_{j=1}^{2} (A^n)_{ij},$$
Which of the following statements is true? (Give the answer without proof)

i. $f(n) = \Theta(\alpha^n)$.

ii. $f(n) = \Omega(n\alpha^n)$.

iii. $f(n) = \mathcal{O}(\alpha^n)$. [2]

(d) Prove the statement that you chose in part (c) of this question. [5]

2. Are the following statements true or false? (Give the answers without proofs)

(a) $n^2 - n - 1 = \Omega(n^2)$. [2]

(b) $n^b = \mathcal{O}(n^a)$, for all $a > b$. [2]

(c) $n^a = \Theta(n^b)$, for all $a \geq b$. [2]

(d) $\log_2(n!) = \Omega(\log_2(n^n))$. [2]

(e) $1 = \mathcal{O}(\sin(n))$. [2]

3. Consider the following MATLAB code. The input to the function is assumed to be a $1 \times n$ matrix, which contains mutually distinct integer numbers.

```
1  function b=fun1(a)
2      n=length(a);
3      for t=1:n-1
4          for i=1:n-1
5              if a(i)>a(i+1)
6                  c=a(i);
7                  a(i)=a(i+1);
8                  a(i+1)=c;
9              end
10         end
11         disp(a)
12     end
13     b=a;
14 end
```

(a) What output will be displayed if you call `fun1([3,4,1])`? [3]

(b) What does `fun1` return when applied to a $1 \times n$ matrix a? [3]

(c) Each time `fun1` passes line 5 it makes one '>' comparison. How many such comparisons does fun1 make when applied to a $1 \times n$ matrix. [3]

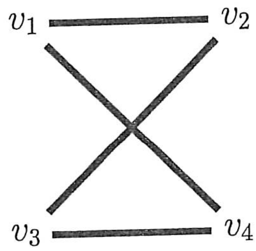
(d) Is `fun1` an efficient algorithm for doing what it does for large n? Justify your answer. [3]

(e) For a 1×5 matrix input, what is the least number of times that `fun1` passes line 6? Given an example of a 1×5 matrix that attains this minimum number. [3]

4. Consider the following MATLAB code. The input to the function is assumed to be an $n \times n$ matrix, which contains integer numbers.

```
function properties=fun2(A)
    n=size(A,1);
    count=0;
    for i=1:n
        d=0;
        for j=1:n
            d=d+A(i,j);
        end
        if 2*floor(d/2)~=d
            count=count+1;
        end
    end
    properties=[False,False];
    if count==0
        properties(1)=True;
    end
    if count==2
        properties(2)=True;
    end
end
```

(a) What output will be returned if you call fun2(A), where A is the adjacency matrix of the graph G, shown below? [3]



(b) List all of the possible outputs that can be returned by calling fun2(A), where A is the 5×5 adjacency matrix of a connected graph on 5 vertices with no self loops. In each case draw an example of a connected graph on 5 vertices with no self loops that gives the corresponding output. [8]

(c) Suppose that G is a connected graph on n vertices with no self loop and let A be the $n \times n$ adjacency matrix of G. Give a geometrical interpretation, in terms of paths and cycles, to the output that would be returned if we called fun2(A). [4]