The idea of this mock exam is to show how this year's open-book policy affects the structure of the test. The first two exercises are the same as last year's. However, the last two exercises are different. The reason being that with the open-book policy one can easily implement the MATLAB code to verify what a function is doing (see last year's pdf for comparison).

1. Let

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{R}$ is a constant such that $\alpha > 0$.

(a) Compute A^2, A^3, A^4.

(b) Prove that

$$A^n = \begin{bmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{bmatrix}$$

for all integers $n \geq 1$

(c) Let

$$f(n) = \sum_{i=1}^{2} \sum_{j=1}^{2} (A^n)_{ij}$$

Which of the following statements is true? (Give the answer without proof)
i. $f(n) = \Theta(\alpha^n)$
ii. $f(n) = \Omega(n\alpha^n)$
iii. $f(n) = \mathcal{O}(\alpha^n)$

(d) Prove the statement that you chose in part (c) of this question.

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2. Are the following statements true or false? (Give the answers without proofs)

(a) $n^2 - n - 1 = \Omega(n^2)$

(b) $n^b = \mathcal{O}(n^a)$, for all $a > b$.

(c) $n^a = \Theta(n^b)$, for all $a \geq b$.

(d) $\log_2(n!) = \Omega(\log_2(n^n))$

(e) $1 = \mathcal{O}(\sin(n))$

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3. Write down in pseudocode an algorithm to compute the binary representation of any integer.

(a) Find the binary representation of 78 .

(b) Find the binary representation of 4098.

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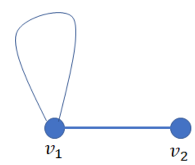
4. Consider the graph G_1 below.



(a) Write down the adjacency matrix of G_1.

(b) For integer $n \geq 1$, how many different walks of length n from v_1 to v_1 does G_1 contain? Justify your answer. You may use theoretical results from the lectures/notes without needing to prove them, but make sure you clearly state any theorem that you apply. Hint: look back at Question 1. Now consider the Wessex vectors w_n, defined by

$$w_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} (w_n)_1 = (w_{n-1})_1 + (w_{n-1})_2 \\ (w_n)_2 = (w_{n-1})_1 + (w_{n-1})_3 \\ (w_n)_3 = (w_{n-1})_2 \end{matrix}$$

for $n \geq 1$

(c) Compute the Wessex vectors w_1, w_2, w_3, w_4 and w_5.

(d) Write down the 3×3 matrix B that satisfies

$$w_n = Bw_{n-1}$$

for $n \geq 1$ and draw the graph G_2 with three vertices $\{v_1, v_2, v_3\}$, which has adjacency matrix B

(e) Prove that G_2 contains exactly ten different walks of length five from v_1 to v_1. Hint: use the fact that $(M)_{11} = (Mw_0)_1$, for any 3×3 matrix M

(f) Hence or otherwise draw a graph G_3 with three vertices $\{v_1, v_2, v_3\}$, that contains exactly six different walks of length two from v_1 to v_1.

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